8. Strength of Components under Combined Loading

8.1. Combination of Two Bendings

8.2. Combination of Bending and Axial Load

8.3. Combined Stress State Analysis

8.4. Combination of Bending and Shear

8.5. Combination of Torsion and Bending

8.1. Combination of Two Bendings
Three-Dimensional Bending Tasks

Bending deformation is present in both component’s principal planes

Structure and its Bending Moment Diagrams

Cantilever Beam

Bending Moment $M_y$ Diagram

Mz diagram

$zx$ Principal Plane

$M_y$ diagram

$xy$ Principal Plane

Loads do not act at the same plane

Bending Moment $M_z$ Diagram

Loads act at the same plane

But this plane is not a principal plane

Internal Forces for Two Bendings’ Case

Bending moments ($M_y$ and $M_z$) act in both principal planes of the component

Shear forces $Q_y$ and $Q_z$ may also be present, but their action is neglected

Reduction of Load(s) to the Principal Axes

Centroid

Critical Cross-Section

Reduction: $F = F_y \cos \alpha$

$F_z = F \sin \alpha$

Bending Moments at the Critical Section

$\begin{align*}
M_y &= F_z L \\
M_z &= F_y L
\end{align*}$

Torque $T$ is also acting at the cross-section

For the cases, when the load direction line does not cross the centroid

Shaft of a Belt Drive

$xy$ Principal Plane

$M_z$ diagram

$M_y$ diagram

Loads do not act at the same plane

Principal Centroidal Axis

Principal Centroidal Axis

Critical Cross-Section
**Bending Stress Analysis**

**Cross-Section' Bending Stresses Diagrams**

- **σ_{My}** diagram
- **σ_{Mz}** diagram

**Equation of NEUTRAL LINE**

\[ \sigma = \frac{M_z}{I_z} y + \frac{M_y}{I_y} z = 0 \]

**Location of Cross-Section Critical Points**

- **Cross-Section NEUTRAL LINE** = projection of the beam' neutral layer
  - line at the beam cross-section, which' points have no stress (bending stress equals to zero)

  **Neutral line divides the section in two: area of tension and area of compression**

**Cross-Section NEUTRAL LINE** = projection of the beam' neutral layer

**Combination of two bendings⇒**

- Neutral Line is straight
- Neutral Line crosses the centroid

**Cross-section critical points are those, that are located most distant from the neutral line**

**Resultant of two bending stresses at some point (y; z) equals to their algebraic sum**

\[ \sigma = \sigma_{My} + \sigma_{Mz} = \frac{M_y}{I_y} z + \frac{M_z}{I_z} y \]

**Resultant of the deformations acting in same direction at some point equals to the sum of these values (considering the signs +/-)**

**According to Hooke's law, the stresses of each cross-section point follow the value of respective strain**

\[ \sigma_{My} = E\varepsilon_{My}, \quad \sigma_{Mz} = E\varepsilon_{Mz} \]
Strength Conditions

Normal Stresses at critical Points

\[
\sigma_{OT} = \frac{M_z}{I_z} y_1 + \frac{M_y}{I_y} z_1 \\
\sigma_{OS} = \frac{M_z}{I_z} y_2 + \frac{M_y}{I_y} z_2
\]

Maximum Tensile Stress at the point \(O_T\)

Maximum Compressive Stress at the point \(O_S\)

Design Stress for Tension

Design Stress for Compression

Strength Conditions

\[
\sigma_{OT} = \frac{M_z}{I_z} y_1 + \frac{M_y}{I_y} z_1 \leq [\sigma]_{\text{Tension}} \\
|\sigma_{OS}| = \left| \frac{M_z}{I_z} y_2 + \frac{M_y}{I_y} z_2 \right| \leq [\sigma]_{\text{Compr}}
\]

Special Case – Rectangular Section

For all cross-sections, which's outside contour is rectangle, both principal axes are the axes of symmetry

Diagonal corners are always critical

Critical points are always located at the diagonal corners of the rectangle

Where in particular the critical points \(O_T\) and \(O_S\) are located, depends on the loading

Value of Maximum Bending Stress

\[
\sigma_{\text{Max}} = \sigma_{\text{Min}} = \frac{|M_y|}{W_y} + \frac{|M_z|}{W_z}
\]

At the critically tensioned point \(O_T\)

At the critically compressed point \(O_S\)
8.2. Combination of Bending and Axial Loads

**Stresses of Circular Section**

- Critical points are always located diametrically on the perimeter of the section.
- Neutral Line is also a principal axis.
- Circle has an infinite number of principal centroidal axes.

**Value of Maximum Normal Stress**

\[ \sigma_{\text{Max}} = \left| \sigma_{\text{Min}} \right| = \frac{|M|}{W} \]

**RESULTANT BENDING MOMENT**

- Can be calculated for circular section only.

**Bending Moment**

\[ M = \sqrt{M_y^2 + M_z^2} \]
Eccentric Tension/Compression

**ECCENTRIC TENSION/COMPRESSION**
- load is acting parallel to the component axis
- load action line does not coincide with axis

**Eccentrically Compressed Bar**
- Short Bar
- Load eccentricity may exceed the bar cross-section dimensions

**Eccentrically Tensioned Bar**

**Short Bar** = Cross-section dimensions and bar length have the same magnitude

Buckling will be studied later

Slender bars have may fail by buckling due to compression

Internal Forces due to Eccentric Load

**Eccentrically Compressed Bar**

Location of the Load

Principal Centroidal Axes

Centroid

**Eccentrically Tensioned Bar**

Location of the Load

Component axis

Centroid

Axial Force $N$ (compression) is shown at both principal planes

**TENSION** ($+$) or **COMPRESSION** ($-$)

**BENDING** may be three-dimensional ($M_y \neq 0$ AND $M_z \neq 0$) or planar ($M_y = 0$ OR $M_z = 0$)

**Cross-Section Internal Forces**

- $N = F (-)$
- $M_y = Fe_z (-)$
- $M_z = Fe_y (-)$

**Internal Forces at the Principal Planes**

- $xz$ Principal Plane
- $xy$ Principal Plane

**Section**

**Axial Force**

$N (-)$

**Bending Moment**

$M_y (-)$

$M_z (-)$

Signs of bending moments $M_y$, $M_z$ depend on the directions of axes $y$ and $z$
Normal Stresses due to Eccentric Load

Axial Stress and Bending Stress are normal stresses, i.e., they both act perpendicular to the cross-section – this is ONE-DIMENSIONAL stress state.

Resultant of an axial and two bending stresses at some point \((y; z)\) equals to their algebraic sum:

\[
\sigma = \sigma_N + \sigma_{My} + \sigma_{Mz} = \frac{N}{A} + \frac{M_y}{I_y} z + \frac{M_z}{I_z} y
\]

Cross-Section Critical Points

Cross-Section NEUTRAL LINE = projection of the beam’ neutral layer

**Neutral line divides the section in two: area of tension and area of compression**

**Equation of NEUTRAL LINE**

\[
\sigma = \frac{N}{A} + \frac{M_y}{I_y} z + \frac{M_z}{I_z} y = 0
\]

The signs (+/-) of both internal forces and coordinates must be considered.

Eccentric Tension/Compression\(\Rightarrow\)

- Neutral Line is straight,
- Neutral Line does not cross the centroid

Cross-Section critical points are the most distant ones from the neutral line.

Critical Points of the Cross-Section

- Maximum Tensile Stress \(O_T (y_1;z_1)\)
- Maximum Compressive Stress \(O_S (y_2;z_2)\)
- Maximum Distance
- Maximum Distance
**Strength Conditions**

### Normal Stresses at the Critical Points

\[
\sigma_{OT} = \frac{N}{A} + \frac{M_z}{I_z} y_1 + \frac{M_y}{I_y} z_1 \\
\sigma_{OS} = \frac{N}{A} + \frac{M_z}{I_z} y_2 + \frac{M_y}{I_y} z_2
\]

Maximum Tensile Stress at the Point \(O_T\)

Maximum Compressive Stress at the Point \(O_S\)

### Strength Conditions for Eccentric Load

\[
\sigma_{OT} = \frac{N}{A} + \frac{M_z}{I_z} y_1 + \frac{M_y}{I_y} z_1 \\ 
|\sigma_{OS}| = \left| \frac{N}{A} + \frac{M_z}{I_z} y_2 + \frac{M_y}{I_y} z_2 \right| \leq \left[ \sigma \right]_{\text{Tension}}
\]

### Stress Distributions of ONE Sign

- The load is applied FAR from the centroid
- The load is applied NEAR to the centroid
- When the load is applied inside the core

**Cross-section’ CORE** = area around the centroid

**Determination of the core** is usually not needed in strength analysis.
**Extremities of Eccentric Loading Cases**

Increasing distance between the load application point and the centroid → More resemblance of the case to that of bending (two bendings) → Neutral line is located nearer to the centroid

**Neutral Line crosses the Centroid**

Load (Force) is applied infinitely distant from the centroid → Nearer the load application point to the centroid → More resemblance of the case to that of axial loading → Increasing distance between the neutral line and the centroid

**Stress Distribution of Two Signs**

Neutral Line is infinitely distant from the Centroid

Load is applied at the centroid

**Special Case – Rectangular Section**

Cross-sections, which' outside contour is rectangle and both principal centroidal axes are axes of symmetry

Critical are the diagonal corners

Critical points are always located at the diagonal corners (or in one corner)

**Maximum Normal Stress Values**

\[
\begin{align*}
\sigma_{\text{Max}} &= \frac{N}{A} + \frac{M_y}{W_y} + \frac{M_z}{W_z} \\
\sigma_{\text{Min}} &= \frac{N}{A} - \frac{M_y}{W_y} - \frac{M_z}{W_z}
\end{align*}
\]
**Special Case – Circular Section**

**Stresses of Eccentrically Loaded circular Section**

- **σ_N diagram**
- **σ_M diagram**
- **σ_M diagram**
- **σ_N diagram**

**Maximum Normal Stress Values**

\[
\sigma_{\text{Max}} = \frac{N}{A} + \frac{M}{W} \\
\sigma_{\text{Min}} = \frac{N}{A} - \frac{M}{W} \\
\]

**Critical points are always located diametrically on the perimeter**

- Resultant bending moment is calculated only for circular sections

**Maximum Bending Moment**

\[
M = \sqrt{M_y^2 + M_z^2}
\]

**Neutral Line**

**This axis is also a principal centroidal axis**

**Circle has an infinite number of principal centroidal axes**

**At the critical point of compression O_s**

**OR**

**At the critical point of tension O_t**

**OR**

**At the critical point of compression O_s**

**OR**

**At the critical point of tension O_t**

**STRENGTH OF MATERIALS**

**8.3. Combined Stress State Analysis**
**Strength Analysis Problem of a Combined Stress State**

**STRENGTH CONDITION** compares the value of actual stress to that of design stress.

**Classical Strength Condition**

\[
\text{ACTUAL STRESS} \leq \text{DESIGN STRESS}
\]

**THREE-DIMENSIONAL** and **ONE-DIMENSIONAL** stresses **CANNOT** be compared.

**PLANE and ONE-DIMENSIONAL** stresses **CANNOT** be compared.

**ACTUAL STRESS**
- Is dependent on the component geometry and loading.
- **ONE-DIMENSIONAL, PLANE or THREE-DIMENSIONAL** stress is acting in the component.

**DESIGN STRESS**
- Is dependent on the material limit state stress.
- Limit state stress is determined by the tensile test.

**ONE-DIMENSIONAL stress** is acting in the test specimen.

**Equivalent Stress of the Combined Stress State**

**KNOWN ARE** the strength conditions for **ONE-DIMENSIONAL** stress state.

**Strength of components, that have PLANЕ or THREE-DIMENSIONAL stress state NEED TO BE ANALYSED**

**COMBINED stress state must be reduced to the ONE-DIMENSIONAL one**

**EQUIVALENT STRESS = ONE-DIMENSIONAL** stress, that has equal criticality with the given **COMBINED** stress state.

**Equivalent Stress of Three-Dimensional Stress State**

\[
\sigma_{Eq} = f(\sigma_1, \sigma_2, \sigma_3) \leq [\sigma]
\]

**Strength Condition for COMBINED Stress State**

**Design stress (Permissible stress) for one-dimensional case**

**Stress states of equal criticality**

**Tensile Test of a Steel**

**Reduction of THREE-DIMENSIONAL Stress State**

**ONE-DIMENSIONAL Stress at the Point K**

**ONE-DIMENSIONAL Stress at the Point K**

**K**

\[\sigma_1, \sigma_2, \sigma_3\]

\[\sigma_{Eq}\]
General Principles of Strength Theories

STRENGTH THEORY (or Limit State Theory) = Theoretical concepts for the stress state criticality analysis

STRENGTH THEORY ⇒
- is based on some hypothesis for limit state appearance: “What kind of stress state causes the material limit state to occur?”
- gives a formula for equivalent stress value calculation

Older
Criteria Theories
- Maximum Normal Stress Theory
- Maximum Strain Theory
- Maximum Shear Stress Theory
- Strain Energy Theory

They give theoretical hypothesis about the general cause of the limit state occurrence (limit state criterion)

Newer
Phenomenological Theories
- Mohr’ Theory, etc

Are based on the mathematical processing results of test data, with no further analysis of physical phenomena

Material Ductility and Brittleness

Limit State of DUCTILE Material = YIELDING
- Elastic Elongation
- Yielding

Limit State of BRITTLE Material = FRACTURE
- Elastic Elongation
- Fracture

Material Ductility and Brittleness depend on TEMPERATURE

Dependence of Steel Ductility on the Temperature
- Malleable Steel
- Ductile
- Brittle

Ductility of many steels start to decrease at the temperatures (0…-10)°C
Theory of Maximum Normal Stress

Applicable for BRITTLE MATERIALS under TENSION

A.K.A. I (first) Strength Theory
W.J.M. Rankine 1820...1872  G. Galilei 1564...1642

Cast Iron, Concrete, Rock, ...

HYPOTHESIS:
Brittle material fractures, when the principal stress of maximum absolute value exceeds a certain limit value, independent on the values of other principal stresses at that point

Ultimate Strength of that material, from the ordinary tensile test

HYPOTHESIS:
Brittle material fractures, when the strain of maximum absolute value exceeds a certain limit value, independent on the values of other strains of that point

Strain, that corresponds to the Ultimate Strength of that material, from the ordinary tensile test

Theory of Maximum Strain

Applicable for BRITTLE MATERIALS under COMPRESSION

A.K.A. II (second) Strength Theory
B. de Saint-Venant 1797...1886  J.V. Poncelet 1788...1867

Cast Iron, Concrete, Rock, ...

HYPOTHESIS:
Brittle material fractures, when the strain of maximum absolute value exceeds a certain limit value, independent on the values of other strains of that point

Strain, that corresponds to the Ultimate Strength of that material, from the ordinary tensile test
**Theory of Maximum Shear Stress**

Applicable for **DUCTILE MATERIALS**, if the limit state is **yielding**

- **H. Tresca**, 1868
- **J.J. Guest**, 1900

**HYPOTHESIS:**

Ductile material starts to yield, when the maximum shear stress of the stress state exceeds a certain limit value, independent of the values of principal stresses at that point.

Maximum shear stress, corresponding to the tensile yield strength of that material, from the ordinary tensile test

- **H. Hencky**, 1925
- **J.J.Guest**, 1900

**Strain Energy Theory**

Applicable for **DUCTILE MATERIALS**, if the limit state is yielding or plasticity

- **R. von Mises**, 1913
- **E. Beltrami**, 1903
- **M.T. Huber**, 1904
- **H. Hencky**, 1925

**HYPOTHESIS:**

Ductile material starts to deform plastically or yield, when the strain energy density of the stress state exceeds a certain limit value.

Strain energy density, that corresponds to the elastic limit stress of that material, from the ordinary tensile test

- **M.T. Huber**, 1904
- **E. Beltrami**, 1903
- **H. Hencky**, 1925
**Mohr’ Strength Theory**

Applicable for both DUCTILE and BRITTLE materials

C.O. Mohr, 1835...1918

Does not have theoretical hypothesis

Phenomenological Theory

From tests with brittle materials

In the cases of combined stress state, the most critical, for the limit state to occur, are the principal stresses of extremal value \( \sigma_1 \) and \( \sigma_3 \) (influence of \( \sigma_2 \) is negligible)

Each three-dimensional stress state can be regarded as a plane stress state

According to Mohr’ Strength Theory

Material Limit State Stresses from Ordinary Tests

\[
\begin{align*}
\sigma_{1\text{Lim}} &= \sigma_U^{\text{Tension}} \\
\sigma_{3\text{Lim}} &= \sigma_U^{\text{Compr}}
\end{align*}
\]

Material tensile Strength form the tensile test

Material Compressive Strength form the compression test

Maximum possible value of \( \sigma_1 \)

Maximum possible value of \( \sigma_3 \)

Value of Equivalent Stress

\[
\sigma_{\text{Eq}} = \sigma_1 - \frac{\sigma_{1\text{Lim}} \cdot \sigma_3}{\sigma_{3\text{Lim}}}
\]

Mohr’ Strength Theory

III Strength Theory

\[
\sigma_{\text{Eq}} = \sigma_{\text{Eq}}^{\text{III}}
\]

Material tensile Strength form the tensile test

Tension

Compr

C.O. Mohr, 1835...1918

Does not have theoretical hypothesis

Phenomenological Theory

From tests with brittle materials

In the cases of combined stress state, the most critical, for the limit state to occur, are the principal stresses of extremal value \( \sigma_1 \) and \( \sigma_3 \) (influence of \( \sigma_2 \) is negligible)

Each three-dimensional stress state can be regarded as a plane stress state

According to Mohr’ Strength Theory

Material Limit State Stresses from Ordinary Tests

\[
\begin{align*}
\sigma_{1\text{Lim}} &= \sigma_U^{\text{Tension}} \\
\sigma_{3\text{Lim}} &= \sigma_U^{\text{Compr}}
\end{align*}
\]

Material tensile Strength form the tensile test

Material Compressive Strength form the compression test

Maximum possible value of \( \sigma_1 \)

Maximum possible value of \( \sigma_3 \)

Value of Equivalent Stress

\[
\sigma_{\text{Eq}} = \sigma_1 - \frac{\sigma_{1\text{Lim}} \cdot \sigma_3}{\sigma_{3\text{Lim}}}
\]

Mohr’ Strength Theory

III Strength Theory

\[
\sigma_{\text{Eq}} = \sigma_{\text{Eq}}^{\text{III}}
\]

General Metod for Combined Stress State Strength Analysis

COMBINED STRESS STATE = any PLANE and THREE-DIMENSIONAL Stress State

Stresses of different directions are acting at the same point

Stresses of different types are acting at the same point

1. Location of critical section is determined using internal force diagrams

2. Location of cross-section critical points is determined using stress diagrams

3. Application of strength condition at the critical point(s) of critical section(s)

Strength Condition by Safety

\[
S = \frac{\sigma_{\text{Lim}}}{\sigma_{\text{Eq}}} \geq [S]
\]

Real Value of Safety Factor

Value of Design Factor

Equivalent one-dimensional stress of the given stress state

Function of stress state’ principal stresses

One-dimensional design stress

There could be many of them

Point, where the equivalent stress maximum value is acting

Points, where some stress maximum value is acting
Equivalent Stress of a Plane Stress State

Principal Stresses of Planar Stress State

According to Maximum Shear Stress’ Strength Theory (Tresca)

\[ \sigma_{\text{E}}^{\text{III}} = \sigma_1 - \sigma_3 \]

According to Strain Energy’ Strength Theory (von Mises)

\[ \sigma_{\text{E}}^{\text{IV}} = \sqrt{\left(\sigma_1^2 + \sigma_3^2 - \sigma_1 \sigma_3\right)} \]

\[ \sigma_{\text{E}}^{\text{IV}} = \sqrt{\sigma_x^2 + \sigma_y^2 - \sigma_x \sigma_y + 3 \tau_{xy}^2} \]

STRENGTH OF MATERIALS

8.4. Combination of Bending and Shear
Internal Forces and Cross-Section Stresses

Combination of Bending and Shear = Interaction of Bending moment $M$ and Shear Force $Q$

Cross-Section Stresses

Bending Stress
$$\sigma_M = \frac{M}{I}$$

Shear Stress
$$\tau_Q = \frac{QS_y}{Ib_y}$$

Stresses of IPE-Section

Combination of Bending and Shear is Important:

Component is relatively short
Shear force $Q$ has high value compared with bending moment $M$

Component has thin walls
Maximums of normal and shear stresses act in close vicinity

Equivalent Stress

Equivalent Stress of a Plane Stress State

According to the maximum Shear Stress Theory (Tresca)
$$\sigma_{Eq}^I = \sigma_1 - \sigma_3$$
$$\sigma_{Eq}^I = \sqrt{\sigma_M^2 + 4\tau_Q^2}$$

According to the Strain Energy Theory (von Mises)
$$\sigma_{Eq}^IV = \sqrt{\sigma_1^2 + \sigma_2^2 - \sigma_1\sigma_2}$$
$$\sigma_{Eq}^IV = \sqrt{\sigma_M^2 + 3\tau_Q^2}$$
Example: Strength Analysis of a IPE-Beam (1)

Determine the appropriate IPE-profile!

Transversely Loaded Beam

Critical Section

For a Combination of Bending and Shear

Maximum bending stress is a function of W, [m²]

Maximum shear stress is a function of A, [m²]

Complex cubic equation is to be solved

Dimensioning for the combination of bending and shear is not easy

Solution

1. Dimensioning for BENDING

2. Check for adequate strength under the combination of bending and shear

Dimensioning for the combination of bending and shear is not easy

Example: Strength Analysis of a IPE-Beam (2)

Dimensioning for BENDING

Strength Condition for Bending

For Moment of Resistance

\[ \sigma_{\text{Max}} = \frac{M}{W} \leq \frac{\sigma_y}{[S]} \]

\[ W \geq \left[ W \right] = \frac{M}{\sigma_y [S]} \]

\[ \left[ W \right] = \frac{M}{\sigma_y [S]} = \frac{6 \cdot 10^3}{235 \cdot 10^6} \cdot 2.5 = 63.8 \cdot 10^{-6} \text{ m}^3 \approx 64 \text{ cm}^3 \]

Table of IPE Steel Profiles

Profile IPE 140 fulfills the strength condition for bending

\[ W_x = 77.3 \text{ cm}^3 \geq \left[ W \right] = 64 \text{ cm}^3 \]

A = 16.4 cm² \( I_x = 541 \text{ cm}^4 \)
Example: Strength Calculation (3)

Check for adequate strength for the COMBINATION OF BENDING AND SHEAR at the critical points of IPE 140 cross-section

Critical Points of the Cross-Section

Critical Points O₃

\[
\begin{align*}
\sigma_{O3} &= 0 \\
\tau_{O3} &= \tau_{Max}
\end{align*}
\]

Critical points O₂

\[
\begin{align*}
\sigma_{O2} &\neq 0 \\
\tau_{O2} &\neq 0
\end{align*}
\]

Critical points O₁

\[
\begin{align*}
\sigma_{O1} &= \sigma_{Max} \\
\tau_{O1} &= 0
\end{align*}
\]

Example: Strength Analysis of a IPE-Beam (4)

Check for adequate strength for BENDING at the critical points O₁ of the IPE 140 cross-section

Maximum bending stress is acting at the critical points O₁

The stress with the same value, but opposite sign is acting at the symmetrically located points

Shear stress is absent

Maximum Bending Stress at the Points O₁

\[
\sigma_{O1} = \frac{M}{W} = \frac{6 \cdot 10^3}{77.3 \cdot 10^{-6}} = 77.6 \cdot 10^6 \text{ Pa} \approx 78 \text{ MPa}
\]

Factor of Safety at the Points O₁

\[
S = \frac{\sigma_y}{\sigma_{Max}} = \frac{235}{78} = 3.01 \geq 3.0 \geq [S] = 2.5
\]

Strength is adequate at the critical points O₁
Example: Strength Analysis of a IPE-Beam (5)

Check for adequate strength for SHEAR at the critical points O₃ of the IPE 140 cross-section

- **Normal stress is absent**
- **Maximum shear stress** is acting at the critical points O₃

Max O₃ \approx 42.9 \text{ MPa}

Check for adequate strength under shear

\[ \tau_{O3} = \frac{QS_{0.5}}{I_\sigma} = \frac{20 \cdot 10^3 \cdot 42.9 \cdot 10^{-6}}{541 \cdot 10^{-8} \cdot 0.0047} = 33.76 \cdot 10^6 \text{ Pa} \approx 34 \text{ MPa} \]

Factor of Safety at the Points O₃

\[ S = \frac{\tau_y}{\tau_{Max}} = \frac{0.5 \cdot 235}{34} = 3.45 \geq 2.5 \]

Strength is adequate at the critical points O₃

Example: Strength Analysis of a IPE-Beam (6)

Check for adequate strength for the COMBINATION of BENDING and SHEAR at the critical points O₂ of the IPE 140 cross-section

- Bending stress value, close to the maximum;
- Shear stress value, close to the maximum.

This is the combination of bending and shear

At the critical points O₂ is acting:

Check for adequate strength under the combination of bending and shear

\[ \sigma_{O2} = \frac{M}{I} \cdot y = \frac{6 \cdot 10^3}{541 \cdot 10^{-8}} \cdot 0.0631 = 69.9 \cdot 10^6 \text{ Pa} \approx 70 \text{ MPa} \]

\[ \tau_{O2} = \frac{QS_{Flange}}{I_\sigma} = \frac{20 \cdot 10^3 \cdot 33.5 \cdot 10^{-6}}{541 \cdot 10^{-8} \cdot 0.0047} = 26.34 \cdot 10^6 \text{ Pa} \approx 27 \text{ MPa} \]
Example: Strength Analysis of a IPE-Beam (7)

Check for adequate strength for the COMBINATION of BENDING and SHEAR at the critical points O₂ of the IPE 140 cross-section

**Equivalent Stress at the Points O₂**

$$\sigma_{Eq, O₂} = \sqrt{\sigma_{O₂}^2 + 4\tau_{O₂}^2} = \sqrt{70^2 + 4 \cdot 27^2} = 88.4 \approx 89 \text{ MPa}$$

**Factor of Safety at the Points O₂**

$$S = \frac{\sigma_y}{\sigma_{Eq}} = \frac{235}{89} = 2.64 \approx 2.6 \geq [S] = 2.5$$

ALL strength conditions are fulfilled

**Strength is adequate at the points O₂**

IPE 140 profile is adequate for the given application

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**STRENGTH OF MATERIALS**

8.5. Combination of Torsion and Bending

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8. Strength of Components under Combined Loading
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8. Strength of Components under Combined Loading
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Combination of Torsion and Bending for the Circular Cross-Section

Stress Distributions

Combination of bending and torsional stress MAXIMUMS is acting at the points O₁ ja O₂

Maximum Bending Stress

\[ \sigma_{M}^{\text{Max}} = \frac{M}{W} = \sqrt{\frac{M_y^2 + M_z^2}{W}} \]

Maximum Torsional Stress

\[ \tau_{T}^{\text{Max}} = \frac{T}{W_0} = \frac{T}{2W} \]

Resultant Bending moment for a CIRCULAR Section

\[ M = \sqrt{M_y^2 + M_z^2} \]

Maximum torsional stresses are acting at the edge of the cross-section

Maximum bending stresses are acting at the edge of the cross-section

This is a plane stress state

Moment of Resistance

Polar Moment of Resistance

Maximum Equivalent Stress for CIRCULAR Section

Equivalent Stress of a Plane Stress State

According to the Maximum Shear Stress Theory (Tresca)

\[ \sigma_{\text{Eq}}^{\text{III}} = \sqrt{(\sigma_x^{\text{Max}})^2 + (\tau_{xy}^{\text{Max}})^2} \]

\[ \sigma_{\text{Eq}}^{\text{III}} = \sqrt{(\sigma_y^{\text{Max}})^2 + (\tau_{xy}^{\text{Max}})^2} + 4\tau_{xy}^2 \]

According to the Strain Energy Theory (von Mises)

\[ \sigma_{\text{Eq}}^{\text{IV}} = \sqrt{(\sigma_x^{\text{Max}})^2 + (\sigma_y^{\text{Max}})^2 - \sigma_x \sigma_y + 3\tau_{xy}^2} \]

\[ \sigma_{\text{Eq}}^{\text{IV}} = \sqrt{(\sigma_M^{\text{Max}})^2 + 3(\tau_T^{\text{Max}})^2} + \sqrt{M_y^2 + M_z^2 + 0.75T^2} \]

Maximum Equivalent Stress for CIRCULAR Section

Stresses

\[ \sigma_x = \sigma_{M}^{\text{Max}} \]

\[ \tau_{xy} = \tau_{T}^{\text{Max}} \]

\[ \sigma_y = 0 \]

At the points O₁ and O₂

According to Maximum Shear Stress Theory (Tresca)

According to the Strain Energy Theory (von Mises)
Equivalent Bending Moment

Maximum Equivalent Stress for a Circular Section
$$\sigma_{\text{Eq}}^{\text{III}} = \sqrt{M_y^2 + M_z^2 + T^2} / W$$
$$\sigma_{\text{Eq}}^{\text{IV}} = \sqrt{M_y^2 + M_z^2 + 0.75T^2} / W$$

Maximum Bending Stress for a Circular Section
$$\sigma_{\text{Eq}} = \frac{M_{\text{Eq}}}{W}$$
or
$$\sigma = \sqrt{M_y^2 + M_z^2} / W$$

Critical section of a uniform circular bar is located, where the equivalent bending moment has maximum value

Maximum Shear Stress Theory (Tresca)
$$M_{\text{Eq}}^{\text{III}} = \sqrt{M_y^2 + M_z^2 + T^2}$$

Strain Energy Theory (von Mises)
$$M_{\text{Eq}}^{\text{IV}} = \sqrt{M_y^2 + M_z^2 + 0.75T^2}$$

Example: Strength Analysis of a Circular Uniform Shaft (1)

Calculate the diameter of this uniform shaft

Shaft of a Belt Drive

Power to be transmitted: $P = 300$ W
Frequency of rotation: $n = 90$ rev/min

Material: steel S355 DIN 17100
$$\sigma_y = R_{ch} = 355 \text{MPa}$$
$$[S] = 2.5$$

Ratio of Belt Sides’ Loads – calculated using Euler’ formula
$$D_1 = 60 \text{ mm}$$
$$D_2 = 100 \text{ mm}$$
$$c = F/f = 2.1$$

General Scheme of Shaft Strength Analysis
1. Bending and Twisting Loads
2. Internal Forces at the Principal Planes
3. Equivalent Bending Moment
4. Strength Calculation in Critical Section(s)
5. Check of Adequate Strength
**Example: Strength Analysis of a Circular Uniform Shaft (2)**

1. **Bending and Twisting Loads**

   **Shaft Angular Velocity**
   
   \[ \omega = \frac{2\pi}{60} = 90 \cdot \frac{2 \cdot \pi}{60} = 9.42 \approx 9.4 \text{ rad/s} \]

   **Moment Transmitted by Shaft**
   
   \[ M = \frac{P}{\omega} = \frac{300}{9.4} = 31.9 \approx 32 \text{ Nm} \]

   **Shaft BENDING deformations are caused by:**
   - Belts' tensioning forces,
   - Transmittable loads, that act transversely

   **Relation of Shaft Twisting Moment and Belts' Forces**
   
   \[ f = \frac{M}{(c-1)D} \]

   **Twisting moment is Caused by the Difference of Belt' Sides' Tensioning Forces**

**Example: Strength Analysis of a Circular Uniform Shaft (3)**

1. **Bending and Twisting Loads (2)**

   - **Tensile Loads of Belts' Sides**
     
     \[ f_1 = 2 \cdot \frac{M}{(c-1)D_1} = 2 \cdot \frac{32}{(2.1-1) \cdot 0.06} = 969.6 \approx 970 \text{ N} \]
     
     \[ f_1 = cf_1 = 2.1 \cdot 970 = 2037 \approx 2040 \text{ N} \]

   - **Shaft Transverse Loads**
     
     \[ F_A = f_1 + f_2 = 2040 + 970 = 3010 \text{ N} \]
     
     \[ F_B = f_2 + f_2 = 1230 + 582 = 1812 \approx 1820 \text{ N} \]

   - **Components of Bending Loads about the Principal Centroidal Axes**
     
     \[ F_{Ay} = F_A = 3010 \text{ N} \]
     
     \[ F_{Az} = 0 \]

   - **Shaft Loads and the Principal Axes**

   - **Components of Transverse Loads**
     
     \[ F_{ay} = F_A \cos \alpha = 1820 \cdot \cos 55^\circ = 1043.9 \approx 1050 \text{ N} \]
     
     \[ F_{az} = F_A \sin \alpha = 1820 \cdot \sin 55^\circ = 1490.8 \approx 1490 \text{ N} \]
Example: Strength Analysis of a Circular Uniform Shaft (4)

2. Internal Forces Diagrams about Principal Centroidal Axes

**Shear forces Q are neglected**

Internal Forces (by the Method of Sections)
\[
\begin{align*}
M_{Cz} &= F_A AC = 3010 \cdot 0.05 = 150.5 \approx 151 \text{ Nm} \\
M_{Dz} &= F_B DB = 1050 \cdot 0.06 = 63 \text{ Nm} \\
M_{Dy} &= F_B DB = 1490 \cdot 0.06 = 89.4 \approx 90 \text{ Nm} \\
T_A &= T_B = M = 32 \text{ Nm}
\end{align*}
\]

Reactions at the bearings can be determined using the equations of equilibrium

NB! In this problem, the calculation of bearing reactions is not needed!!!

Bending Moment Diagram on x-y Plane

Bending Moment Diagram on x-z Plane

Example: Strength Analysis of a Circular Uniform Shaft (5)

3. Diagram of Equivalent Bending Moment

Maximum Shear Stress Theory
\[
M_{\text{Eq}}^{III} = \sqrt{M_y^2 + M_z^2 + T^2}
\]

Values of Equivalent Bending Moment
\[
\begin{align*}
M_{\text{Eq},A}^{III} &= \sqrt{M_A^{2y} + M_A^{2z} + T_A^2} = \sqrt{0^2 + 0^2 + 32^2} = 32 \text{ Nm} \\
M_{\text{Eq},C}^{III} &= \sqrt{M_C^{2y} + M_C^{2z} + T_C^2} = \sqrt{0^2 + 151^2 + 32^2} = 154.3 \approx 155 \text{ Nm} \\
M_{\text{Eq},D}^{III} &= \sqrt{M_D^{2y} + M_D^{2z} + T_D^2} = \sqrt{90^2 + 63^2 + 32^2} = 114.4 \approx 115 \text{ Nm} \\
M_{\text{Eq},B}^{III} &= \sqrt{M_B^{2y} + M_B^{2z} + T_B^2} = \sqrt{0^2 + 0^2 + 32^2} = 32 \text{ Nm}
\end{align*}
\]

This is the critical section of UNIFORM shaft

Maximum equivalent bending moment is acting at the cross-section C

4. Strength Calculation for Cross-Section C

**Strength Condition**
\[
\sigma_{\text{Eq}}^{III} = \frac{M_{\text{Eq}}^{III}}{W} = \frac{32 M_{\text{Eq}}^{III}}{\pi D^3} \leq \frac{\sigma_y}{[S]}
\]

Shaft Diameter for Cross-Section C
\[
D \geq \sqrt[3]{\frac{32 M_{\text{Eq}}^{III}}{\pi \sigma_y [S]}} = \sqrt[3]{\frac{32 \cdot 155}{\pi \cdot 355 \cdot 10^6 \cdot 2.5}} = 0.0223 \text{ m} \approx 25 \text{ mm}
\]
Example: Strength Analysis of a Circular Uniform Shaft (6)

5. Check of Adequate Strength in Cross-Section C

Maximum Bending Stress for Cross-Section C

\[ \sigma_{Mz}^{\text{Max}} = \frac{M_z}{W} = \frac{32 \cdot 155}{\pi D^3} = \frac{101.0}{0.025^3} = 101 \times 10^6 \text{ Pa} \approx 101 \text{ MPa} \]

Maximum Torsional Stress for Cross-Section C

\[ \tau_{Tz}^{\text{Max}} = \frac{T}{W} = \frac{16 \cdot 32}{\pi D^3} = \frac{10.4}{0.025^3} = 11 \text{ MPa} \]

Maximum Equivalent Stress for Cross-Section C

\[ \sigma_{\text{Eq}}^{\text{III}} = \sqrt{\left(\sigma_{Mz}^{\text{Max}}\right)^2 + 4 \left(\tau_{Tz}^{\text{Max}}\right)^2} = \sqrt{101^2 + 11^2} = 103.3 \times 10^6 \text{ Pa} \approx 104 \text{ MPa} \]

Check for Adequate Strength via Equivalent Stress

\[ S = \frac{\sigma_{\text{Eq}}^{\text{III}}}{\sigma_{\text{Eq}}^{\text{Max}}} = \frac{355}{104} = 3.41 \approx 3.4 \geq |S| = 2.5 \]

All strength conditions are fulfilled

Answer: Diameter of that shaft must be 25 mm

Combination of Torsion and Bending for the Rectangular Cross-Section

Three Critical Points:

- Combination of maximum bending stresses
  Corner of a Rectangle
  This is ONE-DIMENSIONAL stress

- Combination of maximum bending stress and maximum torsional stress
  Middle Point of Longer Edge
  This is PLANE stress

- Combination of maximum bending stress and maximum torsional stress
  Middle Point of Shorter Edge
  This is PLANE stress

A point of equal criticality is located symmetrically to the critical points above
Strength Analysis for Rectangular Section

Combination of Normal Stresses at the Point \( O_1 \)

\[
\sigma(O_1) = \sigma_{My}^{\text{Max}} + \sigma_{Mz}^{\text{Max}}
\]

One-Dimensional Stress State

Combination of Normal and Shear Stress at the Points \( O_2 \) and \( O_3 \)

\[
\begin{align*}
\sigma_{\text{Ekv}}(O_2) &= \sqrt{\left(\sigma_{My}^{\text{Max}}\right)^2 + 4\left(\tau_{\text{Tb}}^{\text{Max}}\right)^2} \\
\sigma_{\text{Ekv}}(O_3) &= \sqrt{\left(\sigma_{Mz}^{\text{Max}}\right)^2 + 4\left(\tau_{\text{Tb}}^{\text{Max}}\right)^2}
\end{align*}
\]

Plane Stress States

Strength Calculation for the Most Critical Point

Strength Condition for the Combination of Bending and Torsion

\[
\max[\sigma(O_1), \sigma_{\text{Eq}}(O_2), \sigma_{\text{Eq}}(O_3)] \leq [\sigma]_{\text{Min}}
\]

Minimum Value of the Design Stresses

Strength Condition via Safety Factors

\[
S_{\text{Min}} = \frac{\sigma_{\text{Y}}^{\text{Min}}}{\max[\sigma(O_1), \sigma_{\text{Eq}}(O_2), \sigma_{\text{Eq}}(O_3)]} \geq [S]
\]

Minimum Yield Strength Value

Options of Strength Analysis Methodics

Strength analysis general methodologies depend on the shape of cross-section

Strength analysis using equivalent bending moment

\[
\sigma_{\text{Eq}} = \frac{M_{\text{Eq}}}{W}
\]

Critical points are located somewhere at the circle’s perimeter

Critical points are located at the corners and middle of the rectangle edges

Stop
8.6. Practical Strength Analysis of a Complex Mechanical Structure

Design a Sub-Frame for a Tipper!

Problems

- Complex ladder frame
- Complex loading pattern
- Real loads are not known
- Should be strong enough
- Should be rigid enough
- Should be lightweight enough
- Should be cheap enough

NB! Many requirements are contradictory
New Machine after Some Period of Use:

The box is skewed

The sub-frame is deformed

Rigid Sub-Frame is Reliable and Safe

Hydraulic cylinder axis must be strictly vertical

Hydraulic Cylinder

Tipper Box

Lifting Force $F_{\text{Lift}}$

Tipper box initial position

Superstructure Frame or Sub-Frame
Non-Rigid Sub-Frame Caused the Failure

Hydraulic cylinder axis is inclined about vertical
Tipper Box
Lifting Force $F_{\text{Lift}}$
Hydraulic Cylinder
Lateral Force $F_{\text{Lat}}$
Rear Part of Sub-Frame – Twisted relative to the front
Front Part of Sub-Frame
Veokast liigend raami tagaosas
Hydraulic Cylinder
Axis is inclined about vertical
Tipper Box
Hinge at the Rear of Sub-Frame
Superstructure Frame or Sub-Frame
Tipper Box initial position

Where Was the Mistake Made?

In the Cases of:
• Complexity of structure and
• Complexity of loading

Classical approach of strength analysis is difficult to apply safely

Empirical (based on engineering experience) knowledge must be applied

• General engineering handbooks
• Specific handbooks
• Product catalogues
• Standards (ISO, GOST, DIN, ASME, etc)
• Company standards
• Special methods, etc.
Vehicle Body Builder Instructions were Misunderstood and Violated

STRENGTH OF MATERIALS

THANK YOU!

Questions, please?